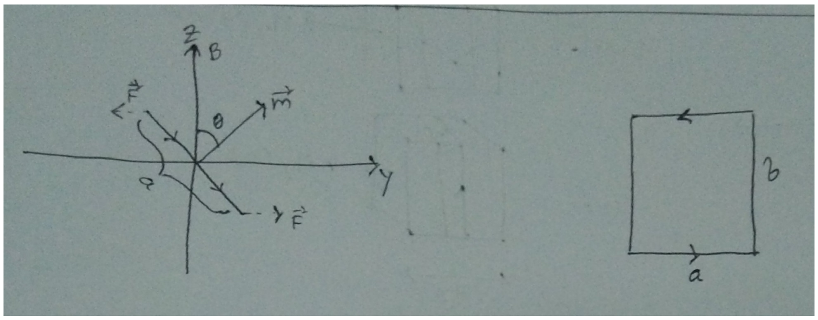
Magnetic Feilds in Matter

Torque on a current carrying loop:

Let us try to calulate the torque on a current carrying loop :



$$\vec{\tau} = \vec{a} \times \vec{F} = a|\vec{F}|\sin\theta\hat{i} \tag{1}$$

while $|\vec{F}| = IB\int dl = IBb$, where b is the breadth of the loop.

$$\begin{aligned} \therefore \vec{\tau} \quad &= aIBb\sin\theta\,\hat{i} \\ &= IabB\sin\theta\,\hat{i} \\ &= mB\sin\,\hat{i} \\ &= \vec{m} \times \vec{B} \end{aligned} \tag{2}$$

were $\vec{m}$ is the dipole moment for the current carrying loop. Thus torque will be minimum if $\theta = 0$. This means that the external magnetic field $\vec{B}$ tries to orient a current carrying loop (dipole) it its own direction. This is understood as the fundamental reason for paramagnetism. We can also calculate the total force on a current carrying loop in a uniform magnetic field :

$$\vec{F} = I\oint d\vec{l} \times \vec{B} = I[\oint d\vec{l}] \times \vec{B} = 0 \tag{3}$$

where $\vec{B}$ can be taken outside the integral since it is a constant and $d\vec{l}$ is a closed loop line integral and hence zero. Thus we see that there will be force on a current carrying loop only in a non-uniform field. We use the generic formula for the force on a volume carrying a current distribution $\vec{J}$

$$\begin{aligned} \vec{F} \quad &= \int \vec{J}(x', y', z') \times \vec{B}(x', y', z') d^3x' \\ &= \int \vec{J}(x', y', z') \times [\vec{B}(0,0,0) + (\vec{x'}.\vec{\nabla}')\vec{B} + \dots] d^3x' \end{aligned} \tag{4}$$

where we have expanded $B_i(\vec{x'}) = B_i(0,0,0) + \vec{x'}.\vec{\nabla}' B_i|_{\vec{x'}=0}$, in a Taylor series. First term is zero as it is equivalent to closed loop integral :

$$\int \vec{J}(x', y', z') \times \vec{B}(0,0,0) d^3x' \sim (\oint I d\vec{l}) \times \vec{B}(0,0,0) = 0 \tag{5}$$

Thus keeping on the first order term we have :

$$\begin{aligned} \vec{F} &= \int \vec{J}(x', y', z') \times (\vec{x'}.\vec{\nabla}')\vec{B}|_{\vec{x'}=0} d^3x' \\ &= \oint I d\vec{l} \times (\vec{x'}.\vec{\nabla}')\vec{B}|_{\vec{x'}=0} d^3x' \end{aligned} \tag{6}$$

Writing it in component form :

$$F_i = \epsilon_{ijk} \oint I \, dl_j (\vec{x'}.\vec{\nabla})\vec{B}_k|_{\vec{x'}=0} \tag{7}$$

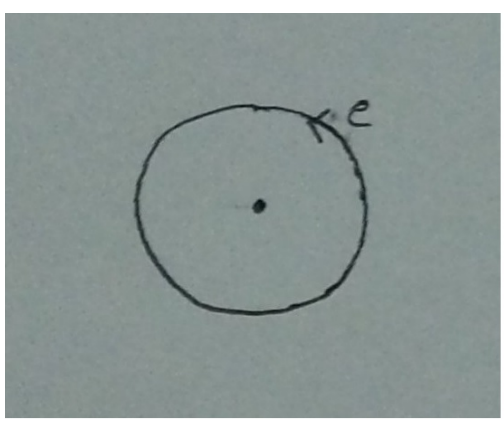
Using vector identity this can be written as :

$$\epsilon_{ijk}(\vec{m} \times \vec{\nabla})_j \vec{B}_k \tag{8}$$

or, $\vec{F} = (\vec{m} \times \vec{\nabla}) \times \vec{B} = \vec{\nabla}(\vec{m}.\vec{B}) - \vec{m}(\vec{\nabla}.\vec{B})$, since $\vec{\nabla}.\vec{B} = 0$ we have : $\vec{F} = \vec{\nabla}(\vec{m}.\vec{B})$. One can also come to the same conclusion by noting that the force on two sides of a rectangular loop will not be same. Hence derive the above in the simpler case.
The above is the reason for para-magnetism as individual atoms can be understood as tiny dipoles trying to orient in the direction of the external magnetic field.

Atomic orbit in a magnetic field

We can think of a simplistic model of an electron around the nucleus (say hydrogen) of an atom.



In this case the current flowing around the "semi-classical" Bohr orbit is given by :

$$I = \frac{q}{T} = \frac{q}{\frac{2\pi R}{v}} = \frac{qv}{2\pi R} \tag{9}$$

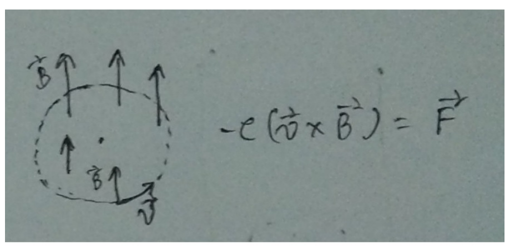
R being the radius of the orbit of the electron, T being the time taken for the electron to go around the orbit once and v is the velocity of the electron. The magnetic moment carried by this electron going about its circular orbit is then given by :

$$|\vec{m}| = I\pi R^2 = \frac{qv}{2\pi R}.\pi R^2 = \frac{qvR}{2} = -\frac{evR}{2} \quad q = -e \text{ for electron} \tag{10}$$

Now the centripetal force is balanced by the Coloumb force in this case:

$$\begin{aligned} \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} &= \frac{m_e v^2}{R} \\ \Rightarrow \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e R} &= v^2 \end{aligned} \tag{11}$$

where m_e is the mass of the electron. Now in the presence of a magnetic field, oriented in direction as shown in the figure :



we have the following balancing equation :

$$\begin{aligned} \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\tilde{v}B &= \frac{m_e\tilde{v}^2}{R} \\ e\tilde{v}B &= \frac{m_e}{R}(\tilde{v}^2 - v^2) \\ e\tilde{v}B &= \frac{m_e}{R}(\tilde{v} - v)(\tilde{v} + v) \end{aligned} \tag{12}$$

here $\tilde{v}$ is the new velocity. assuming $\tilde{v} \sim v$ and $\tilde{v} - v = \Delta v$, is small, we obtain :

$$\begin{aligned} evB &\approx \frac{m_e}{R}\Delta v 2v \\ \Delta v &= \frac{eRB}{2m_e} \end{aligned} \tag{13}$$

This means, since there is a change in the velocity of the electron in orbit, there is also a change in the magnetic moment :

$$\begin{aligned} \Delta|\vec{m}| &= -\frac{1}{2}eR\Delta v \\ &= -\frac{e^2R^2}{4m_e}B \end{aligned} \tag{14}$$

Thus the magnetic moment decreases with increasing magnetic as is clear from the minus sign. One must note that this is a over simplistic model for diamagnetism and the actual reason is much more complicated quantum interaction.

Magnetization

Now we are in a position to study magnetic fields in matter. Since a material object is made of tiny atoms, each atom can be understood as tiny dipoles, which interact with an external magnetic field. In the absence of magnetic field the dipoles are all on average randomly oriented. In the presence of a magnetic field , the dipoles tend to align themselves in the direction of the field (para-magnetism) or in the direction opposite to the field (diamagnetism). This leads to the effect that on an average inside the material, a given volume element will have a net magnetic moment due to orientation of the atomic dipoles in a particular direction. This allows us to define a magnetic moment per unit volume called magnetization.
Recall that the vector potential due to a dipole is given as follows:

$$\vec{A}(\vec{r_1}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{\mathfrak{r}}}{\mathfrak{r}^2} \tag{15}$$

where $\mathfrak{r} = \vec{r_1} - \vec{r_2}$. In this case the we have contribution to the vector potential at $\vec{r_1}$ for infinitesimal magnetic moment $\vec{M}(\vec{r_2})d\tau_2$, where $\vec{M}(\vec{r_2})$ is the magnetic dipole moment per unit volume. We can write contribution from each magnetized volume element as :

$$d\vec{A}(\vec{r_1}) = \frac{\mu_0}{4\pi}\frac{d\vec{m}(\vec{r_2})\times\hat{\mathfrak{r}}}{\mathfrak{r}^2} \tag{16}$$

summing over all such elements we arrive at the integral :

$$\vec{A}(\vec{r_1}) = \frac{\mu_0}{4\pi}\int\frac{\vec{M}(\vec{r_2})d\tau_2\times\hat{\mathfrak{r}}}{\mathfrak{r}^2}, \quad d\vec{m}(\vec{r_2}) = \vec{M}(\vec{r_2})d\tau_2 \tag{17}$$

Noting that :

$$\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} = -\vec{\nabla}_1\frac{1}{|\vec{r_1}-\vec{r_2}|} = \vec{\nabla}_2\frac{1}{|\vec{r_1}-\vec{r_2}|} \tag{18}$$

plugging it in we have:

$$\vec{A}(\vec{r_1}) = \frac{\mu_0}{4\pi}\int\left[\vec{M}(\vec{r_2})\times\vec{\nabla}_2\frac{1}{|\vec{r_1}-\vec{r_2}|}\right]d\tau_2 \tag{19}$$

Now using the vector identity :

$$\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times \vec{\nabla} f \tag{20}$$

we have:

$$\vec{A}(\vec{r_1}) = \frac{\mu_0}{4\pi} \int \frac{1}{\mathfrak{r}} [\vec{\nabla}_2 \times \vec{M}(\vec{r_2})] d\tau_2 - \frac{\mu_0}{4\pi} \int \vec{\nabla}_2 \times \left(\frac{1}{\mathfrak{r}} \vec{M}(\vec{r_2}) \right) d\tau_2 \tag{21}$$

Next we derive another vector identity :

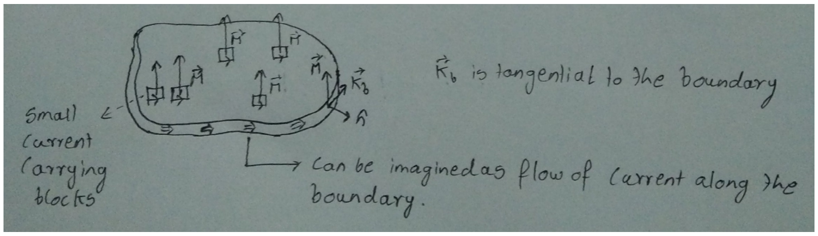
$$\begin{aligned}
&\int_V \vec{\nabla}.\vec{u}\, d\tau = \oint_S \vec{u}.d\vec{s} \quad \text{take } \vec{u} = \vec{A} \times \vec{C} \quad \vec{C} \text{ is a constant vector} \\
&\Rightarrow \int_V \vec{\nabla}.(\vec{A} \times \vec{C}) d\tau = \oint_S (\vec{u}.d\vec{s}).d\vec{s} \\
&\Rightarrow \int_V \vec{C}.(\vec{\nabla} \times \vec{A}) d\tau - \int_V \vec{A}.(\vec{\nabla} \times \vec{C}) d\tau = -\oint_S (\vec{A} \times d\vec{s}).\vec{C}, \quad \vec{\nabla} \times \vec{C} = 0, \text{ since } \vec{C} \text{ is a constant} \\
&\Rightarrow \vec{C}. \int_V (\vec{\nabla} \times \vec{A}) d\tau = -.\vec{C} \oint_S (\vec{A} \times d\vec{s})
\end{aligned} \tag{22}$$

Here V is the volume enclosed and S is the boundary surface of the volume. Since $\vec{C}$ is an arbitrary constant vector, we must have the integrals to be same. Using this we can replace the second term :

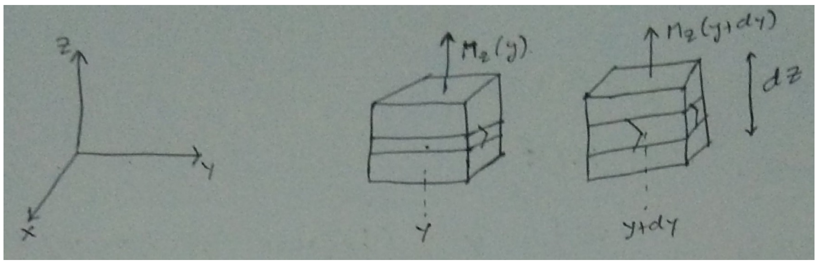
$$\int \vec{\nabla}_2 \times \left(\frac{\vec{M}(\vec{r_2})}{\mathfrak{r}} \right) d\tau_2 = -\oint_S \frac{\vec{M}(\vec{r_2})}{\mathfrak{r}} \times d\vec{s}_2 \tag{23}$$

$$\begin{aligned}
\vec{A}(\vec{r_1}) \quad &= \frac{\mu_0}{4\pi} \int \frac{1}{\mathfrak{r}} [\vec{\nabla}_2 \times \vec{M}(\vec{r_2})] d\tau_2 + \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r_2})}{\mathfrak{r}} \times d\vec{s}_2 \\
&\equiv \frac{\mu_0}{4\pi} \int \frac{\vec{J_b}(\vec{r_2})}{\mathfrak{r}} d\tau_2 + \frac{\mu_0}{4\pi} \int \frac{\vec{K_b}(\vec{r_2})}{\mathfrak{r}} ds_2
\end{aligned} \tag{24}$$

where we define $\vec{J_b}(\vec{r}) = \vec{M}(\vec{r})$ and $\vec{K_b}(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n}(\vec{r})$ where $\hat{n}(\vec{r})$ is the normal to the boundary surface. These are called the "bound" volume and "bound" (denoted by subscript b) surface current densities since they are not generated by actual currents but produced by motion of electrons in their orbitals. We can understand the meaning of $\vec{K_b}$ arising from a surface current flowing in the boundary of a magnetized material .



and $\vec{M}$ a volume current flowing due to non-uniform magnetization. Noting that $|\vec{K}_b| = |\vec{M}|$, we have from the figure :

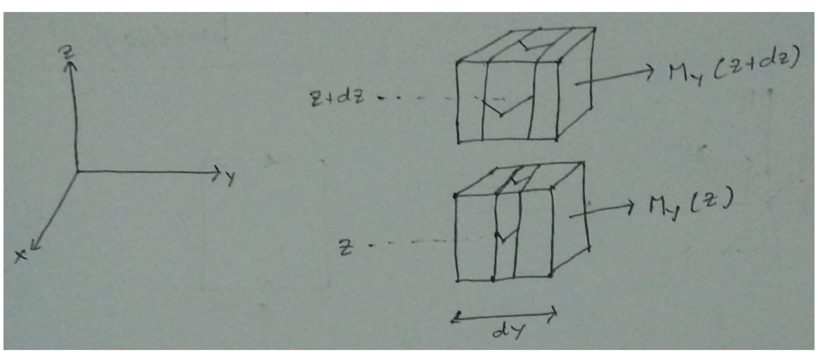


$$
\begin{aligned}
K_{bx}\hat{i} &= M_z\hat{k} \times (-\hat{j}) \text{ at } y + dy \\
K_{bx}\hat{i} &= M_z\hat{k} \times (\hat{j}) \text{ at } y
\end{aligned}
\tag{25}
$$

Thus the current due to this segment is given by :

$$
I_{x1} = [M_z(y + dy) - M_z(y)]dz = \frac{\partial M_z}{\partial y} dy\, dz \tag{26}
$$

similarly from another configuration :



$$
I_{x2} = [-M_y(z + dz) + M_y(z)]dy = -\frac{\partial M_y}{\partial z} dz\, dy \tag{27}
$$

Adding th two contributions :

$$I_{x,total} = [\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}]dzdy \tag{28}$$

Thus we find the x component of volume current density in this case :

$$J_x = \frac{I_x}{dz\,dy} = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} = \vec{\nabla} \times \vec{M}|_x \tag{29}$$

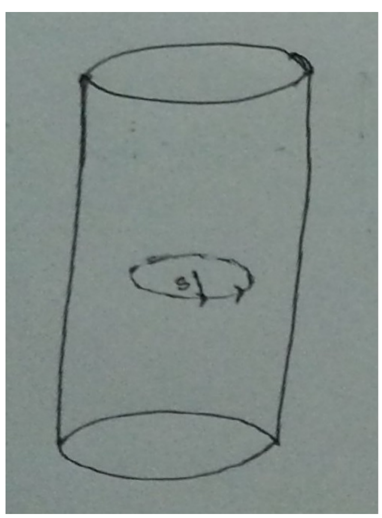
evidently we have in this case : $\vec{\nabla}.\vec{J} = \vec{\nabla}.(\vec{\nabla} \times \vec{M}) = 0$.
Thus we have the total current density : $\vec{J} = \vec{J_b} + \vec{J_f}$, here f in the subscript means free current density. In this case we have :

$$\begin{aligned} &\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = \mu_0(\vec{J_b} + \vec{J_f}) = \mu_0(\vec{\nabla} \times \vec{M} + \vec{J_f}) \\ &\Rightarrow \vec{\nabla} \times (\frac{\vec{B}}{\mu_0} - \vec{M}) = \vec{J_f} \end{aligned} \tag{30}$$

Define the Auxiliary field / Magnetizing field $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$, we have :

$$\begin{aligned} &\vec{\nabla} \times \vec{H} = \vec{J_f} \quad \text{applying Stokes theorem} \\ &\int_S \vec{H}.d\vec{s} = \oint \vec{H}.d\vec{l} = \int_S \vec{J_f}.d\vec{s} = I_f, \text{ analog of Ampere's law} \end{aligned} \tag{31}$$

Thus the field $\vec{H}$ is generated by free currents only. Let us do an example for the case of total current I flowing uniformly through a cylindrical wire of radius R and find the Auxiliary field in that case. Using the integral above for a circular loop around the axis of the cylinder:



$$
\begin{aligned}
\int \vec{H}.d\vec{l} = \int_0^{2\pi} H_\phi \hat{\phi}.s\, d\phi\hat{\phi} = H_\phi 2\pi s \qquad &= I \quad s > R \\
&= \frac{I}{\pi R^2}\pi s^2 \quad s < R
\end{aligned} \tag{32}
$$

Hence we have :

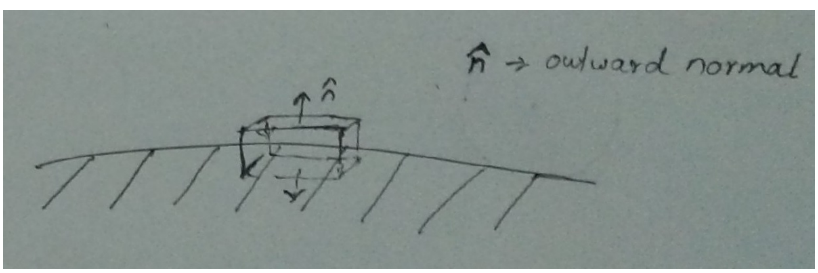
$$
\begin{aligned}
H_\phi \qquad &= \frac{I}{2\pi s} \quad s > R \\
&= \frac{Is}{2\pi R^2} \quad s < R
\end{aligned} \tag{33}
$$

Boundary conditions

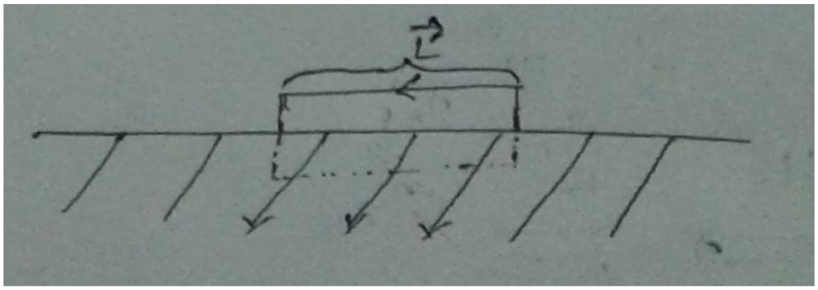
Just like in electrostatics we have conditions that the field satisfies at the boundary of two regions, in this case also we have the same. Noting that $\vec{B} = \mu_0(\vec{M}+\vec{H})$, applying divergence on both sides yields :

$$
\begin{aligned}
&\vec{\nabla}.\vec{B} = \mu_0(\vec{\nabla}.\vec{M} + \vec{\nabla}.\vec{H}) = 0 \\
&\Rightarrow \vec{\nabla}.\vec{M} + \vec{\nabla}.\vec{H} = 0
\end{aligned} \tag{34}
$$

From the above we can use Gauss divergence theorem to find the boundary conditions of $\vec{H}$ and $\vec{M}$. Taking a Gaussian pill box of infinitesimal thickness on the boundary of a magnetized material say :



$$
\begin{aligned}
&\int_V dV\, \vec{\nabla}.\vec{H} = -\int_V dV\, \vec{\nabla}.\vec{M} \\
&\Rightarrow \oint_S d\vec{s}.\vec{H} = -\oint_S d\vec{s}.\vec{M} \\
&\Rightarrow \vec{H}.\hat{n}|_{\text{above}} - \vec{H}.\hat{n}|_{\text{bellow}} = -\vec{M}.\hat{n}|_{\text{above}} - \vec{M}.\hat{n}|_{\text{bellow}} \\
&\Rightarrow H^\perp_{\text{above}} - H^\perp_{\text{bellow}} = -M^\perp_{\text{above}} + M^\perp_{\text{bellow}}
\end{aligned} \tag{35}
$$



while using a loop of infinitesimal width we have :

$$\begin{aligned}
&\vec{\nabla} \times \vec{H} = \vec{J}_f \\
&\oint_{\perp} \vec{H}.d\vec{l} = I_f \\
&\vec{H}_{above}.\vec{l} - \vec{H}_{bellow}.\vec{l} = Kl, \quad \vec{K} \text{ is the surface current flowing along the boundary}
\end{aligned} \tag{36}$$

if the loop is perpendicular to the flow of surface current $\vec{K}$. While :

$$\begin{aligned}
&\oint_{||} \vec{H}.d\vec{l} = 0 \\
&\vec{H}_{above}.\vec{l} - \vec{H}_{bellow}.\vec{l} = 0
\end{aligned} \tag{37}$$

if the loop is parallel to the direction of flow of surface current $\vec{K}$. This means :

$$\vec{H}^{||}_{above} - \vec{H}^{||}_{bellow} = \vec{K} \times \hat{n} \tag{38}$$

$\hat{n}$ is the outward normal to the surface. We immediately see that if $\vec{l}\,||\,\vec{K}$ then $(\vec{K} \times \hat{n}).\vec{l} = 0$ and if $\vec{l} \perp \vec{K}$ we have : $(\vec{K} \times \hat{n}).\vec{l} = Kl$.

The magnetic scalar potential and its applications

If we have a region where there is no free current then : $\vec{\nabla} \times \vec{H} = 0$, then we can do the following replacement (analog of electrostatics) : $\vec{H} = -\vec{\nabla}.\Phi_M$. Thus we have : $-\vec{\nabla}.\vec{\nabla}\Phi_M = -\vec{\nabla}.\vec{M}$, which is Poisson's equation. Φ_M is called the magnetic vector potential. Let us do an example to see where this could be useful. If we have a sphere of constant magnetization $M\hat{z}$, radius R, then this is a scenario where we can use the magnetic scalar potential since there is no free current. Since $\vec{M}$ is constant inside and zero outside, in both cases we have $\vec{\nabla}.\vec{M} = 0$. We use the solution of the Laplace equation inside the sphere and outside depending on asymptotic at $r \to 0$ and $r \to \infty$, such that the solutions be non-singular at these limits. Thus :

$$\Phi_M(\text{inside}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad (r < R)$$
$$\Phi_M(\text{outside}) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad (r > R) \tag{39}$$

Now from continuity of the vector potential we have:

$$\Phi_M(\text{inside}) = \Phi_M(\text{outside}) \qquad \text{at } r = R \text{ for all } \theta \tag{40}$$

Yielding from orthogonality of $P_l(\cos\theta)$, $\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}\delta_{l,l'}$:

$$A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow B_l = R^{2l+1} A_l \tag{41}$$

again :

$$\begin{aligned}
&\left[(\vec{H}.\hat{n})_{\text{outside}} - (\vec{H}.\hat{n})_{\text{inside}}\right]_{r=R} = -\left[(\vec{M}.\hat{n})_{\text{outisde}} - (\vec{M}.\hat{n})_{\text{inside}}\right]_{r=R} \\
&\Rightarrow -(\hat{r}.\vec{\nabla}\Phi_M)_{\text{outside}} + (\hat{r}.\vec{\nabla}\Phi_M)_{\text{outside}} = (\vec{M}.\hat{r})_{\text{inside}} \\
&\Rightarrow -\left[\left(\frac{\partial \Phi_M}{\partial r}\right)_{\text{outside}} + \left(\frac{\partial \Phi_M}{\partial r}\right)_{\text{inside}}\right]_{r=R} = M\hat{z}.\hat{r} = M\cos\theta
\end{aligned} \tag{42}$$

Where we note that in this case $\hat{n} = \hat{r}$ since the boundary is the surface of the sphere. Plugging in the expansions we obtain :

$$\begin{aligned}
&\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) + \sum_{l=1}^{\infty} lA_l R^{l-1}P_l(\cos\theta) = M\cos\theta \\
&\Rightarrow \sum_{l=0}^{\infty}(l+1)A_l R^{l-1}P_l(\cos\theta) + \sum_{l=1}^{\infty} lA_l R^{l-1}P_l(\cos\theta) = M\cos\theta \\
&\Rightarrow A_0 R^{-1}P_0(\cos\theta) + \sum_{l=1}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta) = M\cos\theta
\end{aligned} \tag{43}$$

Using orthogonality of $P_l(\cos\theta)$, and that $\cos\theta = P_1(\cos\theta)$:

$$\begin{aligned}
&A_l = 0, \quad l \neq 1 \\
&3A_1 P_1(\cos\theta) = MP_1(\cos\theta) \\
&\Rightarrow A_1 = \frac{M}{3}
\end{aligned} \tag{44}$$

Hence we get :

$$\begin{aligned}
&\Phi_M(inside) = \frac{M}{3} r\cos\theta = \frac{M}{3} z \\
&\vec{H} = -\vec{\nabla}\Phi_M = -\frac{M}{3}\hat{z} \\
&\vec{B} = \mu_0(\vec{M}+\vec{H}) = \mu_0(M\hat{z} - \frac{M}{3}\hat{z}) = \frac{2}{3}\mu_0 M\hat{z} = \frac{2}{3}\mu_0(\frac{4}{3}\pi R^3\vec{M})\frac{1}{(\frac{4}{3}\pi R^3)} = \frac{2}{3}\mu_0\vec{m}\frac{1}{V}
\end{aligned} \tag{45}$$

where $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$ is the total magnetic moment of the magnetized sphere. While for the region outside:

$$\begin{aligned}
\Phi_M(outside) &= \frac{B_1}{r^2}P_1(\cos\theta) = \frac{R^3 M}{3}\frac{1}{r^2}\cos\theta \\
\vec{H} &= -\vec{\nabla}\Phi_M = -\left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\Phi_M \\
&= \frac{\hat{r}}{4\pi}\left[\frac{4\pi R^3}{3}M\right]\frac{2\cos\theta}{r} + \frac{\hat{\theta}}{4\pi}\left[\frac{4\pi R^3}{3}M\right]\frac{\sin\theta}{r} \\
\vec{B} &= \frac{\mu_0 m}{4\pi}\left[\hat{r}\frac{2\cos\theta}{r^3} + \hat{\theta}\frac{\sin\theta}{r^3}\right]
\end{aligned} \tag{46}$$

Hence outside the sphere we get the field due to a a dipole with total dipole moment $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$. Thus :

$$\begin{aligned}
\vec{B} \quad &= \frac{2}{3}\mu_0\vec{m}\frac{1}{V} \qquad r < R \\
&= \frac{\mu_0 m}{4\pi}\left[\hat{r}\frac{2\cos\theta}{r^3} + \hat{\theta}\frac{\sin\theta}{r^3}\right] \qquad r > R
\end{aligned} \tag{47}$$

In the limit $V \to 0$ limit with $MV = m$ constant, we obtain :

$$\vec{B} = \frac{\mu_0 m}{4\pi}\left[\hat{r}\frac{2\cos\theta}{r^3} + \hat{\theta}\frac{\sin\theta}{r^3}\right] + \frac{2}{3}\mu_0\vec{m}\delta^{(3)}(\vec{r}) \tag{48}$$

Magnetic Suceptibility

As discussed in Paramagnetic effect and Diamagnetic effect we have $\vec{M} \propto \vec{B}$. We can also write this in terms of $\vec{H}$. This is called a linear magnetic material. $\vec{M} = \chi_m\vec{H}$, where χ_m is called the magnetic susceptibility. Plugging this in the equation for $\vec{B}$, we obtain :

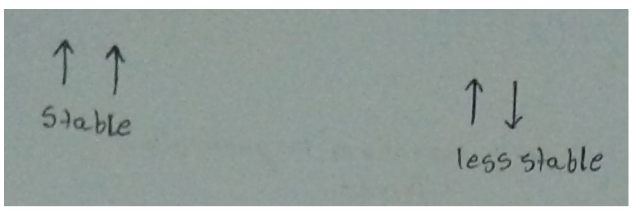
$$\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0(\vec{H}+\chi_m\vec{H}) = \mu_0(1+\chi_m)\vec{H} = \mu\vec{H} \tag{49}$$

where μ is defined as the permeability of the material. Evidently where there is not Magnetization i.e. in vacuum , $\chi_m = 0$.

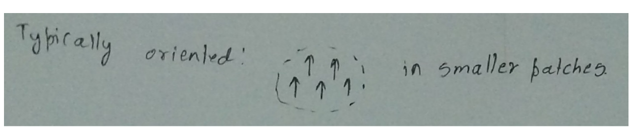
Ferromagnetic

We shall now briefly discuss about Ferromagnetism.

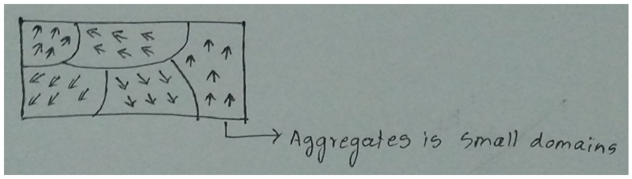
- This arises from fundamental property of electron, its magnetic dipole moment, as it behaves like a tiny magnet producing a magnetic field.
- The dipole moment comes from the fundamental property of the electron → its quantum mechanical spin. The electron spin can have two possible choices : up (spin $\frac{1}{2}$) and down (spin $-\frac{1}{2}$).
- Also orbital motion produces dipole moment, but that is less.
- Now atoms which have filled electron shells have total dipole moment zero, because electrons all exist in pairs with opposite spins, due to Pauli exclusion principle. Thus everywhere magnetic moment is canceled.
- Atoms which have free electrons in orbitals. Have essentially tiny dipoles and tend to align parallel to an external magnetic field.
- Two nearby atoms having unpaired electrons interact :
  → If the spins (dipoles) are parallel then the configuration has lesser energy
  → If the spins (dipoles) are anti-parallel then the configuration has more energy



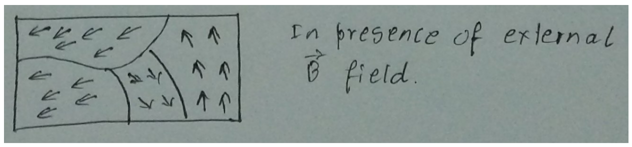
- Typically the spins are alligned in patches which aggregrate together, called "domains".

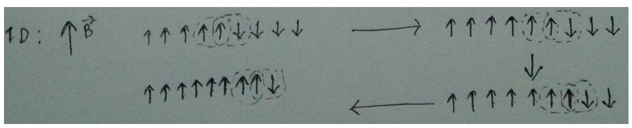


But in the absence of any external magnetic field these domains are randomly oriented, such that the net magnetization $\sim$ zero after avergaing over all the domains, as one domain kills the effect of the other.

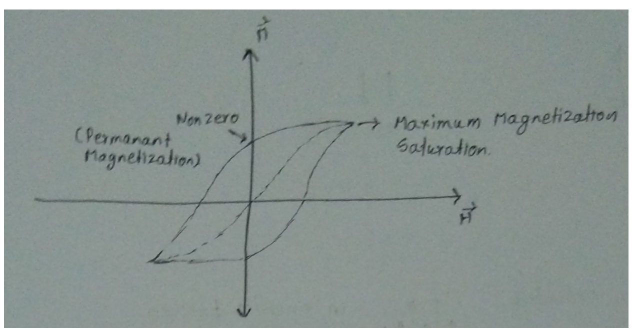


- Now as one switches on a magnetic field $\vec{B}$, Domains with dipole parallel increases in size as due to effect of torque on dipoles, which is dominant on the boundaries as one has competing neighbouring spin interactions .





- But now if $\vec{B} \to 0$ then domains do not go back to the original configuration. Hence there is residual magnetization left. In fact the Magnetization follows a curve with applied field ($\vec{H}$), noting in this case $\vec{M} >> \vec{H}$ and hence $\vec{B} \approx \vec{M}$



Examples

1 Energy of a magnetic dipole in a magnetic field :

$$\begin{aligned}
\vec{F} &= \vec{\nabla}(\vec{m}.\vec{B}) \\
U &= -\int_{\infty}^{r} \vec{F}.d\vec{l} \\
&= -\int_{\infty}^{r} \vec{\nabla}(\vec{m}.\vec{B}).d\vec{l} \\
&= -\vec{m}.\vec{B}(\vec{r}) + \vec{m}.\vec{B}(\infty)
\end{aligned} \tag{50}$$

Assuming $\vec{B}(\infty) = 0$, we obtain $U(\vec{r}) = -\vec{m}.\vec{B}(\vec{r})$. Another way to do the problem is to realize that the work done on the dipole due to the magnetic field $\vec{B}$ is given by :

$$\begin{aligned}
dW &= -\vec{\tau}.d\vec{\theta} \\
&= -mB\sin\theta d\theta
\end{aligned} \tag{51}$$

Where the $-$ sign is because of work done on the system.
2 Using this one can find the interaction energy due to two dipoles of moments $\vec{m}_1$ and $\vec{m}_2$ separated from one another by displacement vector $\vec{r}$. Taking say dipole $\vec{m}_1$ at origin, the magnetic field at point $\vec{r}$ where the dipole $\vec{m}_2$ is situated is given by :

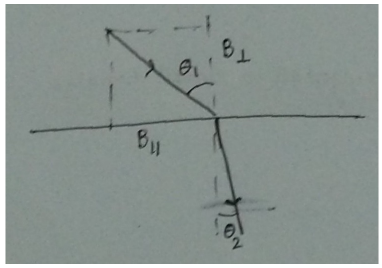
$$\vec{B}_{\vec{m}_1} = \frac{\mu_0}{4\pi r^3}[3(\vec{m}_1.\hat{r})\hat{r} - \vec{m}_1] \tag{52}$$

The potential energy of the dipole $\vec{m}_2$ in this field is given by :

$$U = -\vec{m}_2.\vec{B}_{\vec{m}_1} = -\frac{\mu_0}{4\pi r^3}[3(\vec{m}_1.\hat{r})(\hat{r}.\vec{m}_2) - \vec{m}_1.\vec{m}_2] \tag{53}$$

Evidently this is symmetric in $\vec{m}_1$ and $\vec{m}_2$.

3 Given two mediums with absolute permeability μ_{above} and μ_{bellow}, the magnetic lines of force are shown as follows:



To find the ratio of the slopes of the lines of forces we use the boundary conditions :

$$B^{\perp}_{above} - B^{\perp}_{bellow} = 0 \tag{54}$$

Which comes from $\vec{\nabla}.\vec{B} = 0$. More over we have:

$$\vec{H}^{||}_{above} - \vec{H}_{bellow} = \vec{K}_f \times \hat{n} \tag{55}$$

Note that in this case $\vec{K}_f = 0$ and $\vec{B}_{\text{above/bellow}} = \mu_{\text{above/bellow}}\vec{H}_{\text{above/bellow}}$. Hence:

$$\vec{B}^{||}_{\text{above/bellow}} = \mu_{\text{above/bellow}}\vec{H}^{||}_{\text{above/bellow}} \tag{56}$$

Hence :

$$\begin{aligned} \tan\theta_1 &= \frac{B^{||}_{above}}{B^{\perp}_{above}} \\ \tan\theta_2 &= \frac{B^{||}_{bellow}}{B^{\perp}_{bellow}} \end{aligned} \tag{57}$$

taking ratio of the two :

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{B^{||}_{\text{above}}}{B^{\perp}_{\text{above}}}\frac{B^{\perp}_{\text{bellow}}}{B^{||}_{\text{bellow}}} = \frac{\mu_{\text{bellow}}H^{||}_{\text{bellow}}}{\mu_{\text{above}}H^{||}_{\text{above}}} = \frac{\mu_{\text{bellow}}}{\mu_{\text{above}}} \tag{58}$$